§ Covariant derivatives (do Carmo §4.4)

Given a surface $S \subseteq \mathbb{R}^3$, recall that a vector field on S

$$X : S \longrightarrow \mathbb{R}^3 \qquad \text{(smooth)}$$

- is tangential if $X_p \in T_pS \quad \forall\, p \in S$
- is normal if $X_p \in (T_pS)^\perp \quad \forall\, p \in S$

Def$^{\underline{n}}$: $\mathfrak{X}(S) := \{\text{tangential vector fields on } S\}$

$\mathfrak{X}^\perp(S) := \{\text{normal vector fields on } S\}$

Q: How to differentiate vector fields in $\mathfrak{X}(S)$?

A: Covariant derivatives!

Def$^{\underline{n}}$: Given $X, Y \in \mathfrak{X}(S)$, define the covariant derivative of Y along X as

$$\boxed{\nabla_X Y := (D_X Y)^T}$$

where $(\cdot)^T$ refers to the tangential component of a vector based at $p \in S$ according to the orthogonal splitting

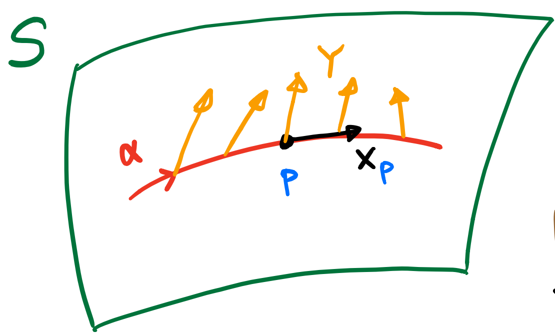
$$\left(\text{depends on } p\right) \quad \boxed{\begin{array}{ccccc} (\mathbb{R}^3 =) T_p\mathbb{R}^3 & = & T_pS & \oplus & (T_pS)^\perp \\ \Downarrow & & \Downarrow & & \Downarrow \\ v & = & v^T & + & v^\perp \end{array}} \quad (*)$$

Remarks: (1) Recall that $D_XY(p)$ depends <u>ONLY</u> on

(a) the vector X_p

and (b) the values of Y restricted to <u>ANY</u> curve

$\alpha : (-\varepsilon, \varepsilon) \to \mathbb{R}^3$ s.t. $\alpha(0) = p$, $\alpha'(0) = X_p$



$$D_XY(p) = \left.\frac{d}{dt}\right|_{t=0} Y(\alpha(t))$$

Hence, this is well-defined even X, Y are only defined on S.

(2) $X, Y \in \mathfrak{X}(S) \Rightarrow \nabla_XY \in \mathfrak{X}(S)$

We now study some important properties of ∇.

Properties of ∇: Let $X, Y, Z \in \mathfrak{X}(S)$, $f \in C^\infty(S)$,

a, b are real constants.

(1) Linearity in both variables:

$$\nabla_X (aY + bZ) = a\nabla_X Y + b\nabla_X Z$$

$$\nabla_{aX+bY} Z = a\nabla_X Z + b\nabla_Y Z$$

(2) Liebniz rule: $\nabla_X (fY) = X(f)\,Y + f\nabla_X Y$

(3) Tensorial: $\nabla_{fX} Y = f\nabla_X Y$

(4) Torsion free: $\boxed{\nabla_X Y - \nabla_Y X = [X, Y]}$

(5) Metric compatibility:

$$\boxed{X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle X, \nabla_X Z\rangle}$$

Remark: The covariant derivative

$$\nabla : \mathfrak{X}(S) \times \mathfrak{X}(S) \longrightarrow \mathfrak{X}(S)$$

$$X\ ,\ Y \longmapsto \nabla_X Y$$

is uniquely defined by properties (1) – (5) above!

"Fundamental Theorem of Riemannian geometry"

Proof: It follows from the fact that (1) - (5) are satisfied with "∇" replaced by "D" for vector fields in $\mathbb{R}^3$.

E.g. To prove (5),

$$X\langle Y, Z\rangle = \langle D_X Y, Z\rangle + \langle Y, D_X Z\rangle$$

$$(\text{since } Y, Z \in \mathfrak{X}(S)) \quad = \langle \underbrace{(D_X Y)^T}_{= \nabla_X Y}, Z\rangle + \langle Y, \underbrace{(D_X Z)^T}_{= \nabla_X Z}\rangle$$

$\square$

§ Einstein summation convention

- 2 kinds of indices: upper & lower
- Same index appearing BOTH as an upper & lower index in the same term $\Longrightarrow$ Sum over this index

E.g: coordinates in $\mathbb{R}^n$: $x^1, x^2, \dots, x^n$ (upper)

coordinate vector fields in $\mathbb{R}^n$: $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}$ (lower)

vector fields in $\mathbb{R}^n$: $X = \sum_{i=1}^{n} a^i \frac{\partial}{\partial x^i} = a^i \frac{\partial}{\partial x^i}$

Einstein summation convention

E.g. $\mathbf{X}(u^1, u^2) : \mathcal{U} \to S \subseteq \mathbb{R}^3$ parametrization

Write $\boxed{\partial_i := \frac{\partial \mathbf{X}}{\partial u^i}}$ coordinate vector fields

Any tangential vector field $X \in \mathfrak{X}(S)$ can be locally expressed as $X = a^i \partial_i = a^k \partial_k$

"dummy index"

<u>1^{st} f.f</u>: $g_{ij} = \langle \partial_i, \partial_j \rangle$

<u>inverse</u>: $(g^{ij}) = (g_{ij})^{-1}$

<u>2^{nd} f.f.</u>: $A_{ij} = \left\langle \frac{\partial^2 \mathbf{X}}{\partial u^i \partial u^j}, N \right\rangle$

E.g. $\langle a^i \partial_i, b^j \partial_j \rangle = g_{ij} a^i b^j$

$g^{ik} g_{kj} = \delta_{ij}$

§ Christoffel Symbols (do Carmo § 4.3)

Given a parametrization $\mathbf{X}(u^1, u^2)$ on S

tangent vectors: $\partial_1 = \dfrac{\partial \mathbf{X}}{\partial u^1}$, $\partial_2 = \dfrac{\partial \mathbf{X}}{\partial u^2}$

unit normal: $N = \dfrac{\partial_1 \times \partial_2}{\| \partial_1 \times \partial_2 \|}$

At each point $p \in S$, we have the orthogonal splitting:

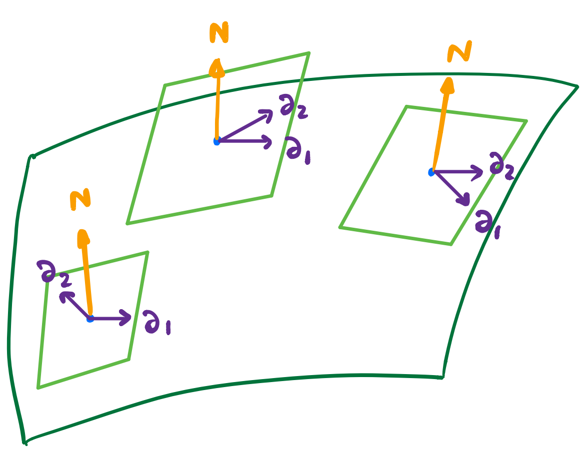
$$\mathbb{R}^3 \cong T_p\mathbb{R}^3 = \underbrace{T_pS}_{\boxed{\partial_1, \partial_2}} \oplus \underbrace{(T_pS)^{\perp}}_{N}$$

basis: $\boxed{\partial_1, \partial_2} \perp N$

Caution: NOT orthonormal!

$(g_{ij}) = \langle \partial_i, \partial_j \rangle \neq \delta_{ij}$

We have a 2-parameter family of basis of $\mathbb{R}^3$ along S:



Similar to a "moving frame", but NOT orthonormal!

<u>Def$^{\underline{n}}$</u>: The Christoffel symbols Γ_{ij}^{k} of a given coordinate system u^1, u^2 is defined as

$$\boxed{\nabla_{\partial_i} \partial_j = \Gamma_{ij}^{k} \partial_k}$$

<u>Prop</u>: (1) $\Gamma_{ij}^{k} = \Gamma_{ji}^{k}$ (symmetry)

(2) $$\boxed{\Gamma_{ij}^{k} = \frac{1}{2} g^{k\ell} \left(\partial_i g_{\ell j} + \partial_j g_{i\ell} - \partial_\ell g_{ij} \right)}$$ Koszul formula

<u>Proof</u>: (1) Since ∇ is torsion-free, i.e.

$$\underbrace{\nabla_{\partial_i} \partial_j - \nabla_{\partial_j} \partial_i}_{\substack{\parallel \\ (\underbrace{\Gamma_{ij}^{k} - \Gamma_{ji}^{k}}_{=0}) \partial_k}} = [\partial_i, \partial_j] = 0$$

(2) Since ∇ is metric compatible,

$$X \langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle$$

for any tangential vector fields $X, Y, Z \in \mathfrak{X}(S)$

In particular, if we choose $X = \partial$, $Y = \partial_i$, $Z = \partial_j$

$$\Rightarrow \quad \partial_\ell g_{ij} = \langle T^k_{\ell i}\, \partial_k , \partial_j \rangle + \langle \partial_i , T^k_{\ell j}\, \partial_k \rangle$$

$$= T^k_{\ell i}\, g_{kj} + T^k_{\ell j}\, g_{ik}$$

Cyclicly permuting i, j, ℓ, we obtain using (1)

$$\begin{aligned}
- \quad \partial_\ell g_{ij} &= \cancel{T^k_{\ell i}\, g_{kj}} + \cancel{T^k_{\ell j}\, g_{ik}} \\
+ \quad \partial_i g_{j\ell} &= T^k_{ij}\, g_{k\ell} + \cancel{T^k_{i\ell}\, g_{jk}} \\
+ \quad \partial_j g_{\ell i} &= \cancel{T^k_{j\ell}\, g_{ki}} + T^k_{ji}\, g_{\ell k} \\
\hline
\partial_i g_{\ell j} + \partial_j g_{i\ell} - \partial_\ell g_{ij} &= 2\, g_{k\ell}\, T^k_{ij}
\end{aligned}$$

Multiplying by $g^{m\ell}$ on both sides

$$g^{m\ell}\left(\partial_i g_{\ell j} + \partial_j g_{i\ell} - \partial_\ell g_{ij}\right) = 2\, \underbrace{g^{m\ell}\, g_{k\ell}}_{= \delta^m{}_k}\, T^k_{ij} = 2\, T^m_{ij}$$

Dividing **2** on both sides and switch **m** to **k** gives the desired formula.

□